TRINITY COLLEGE

IN NOMINE DOMINI

Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (50 Marks)**

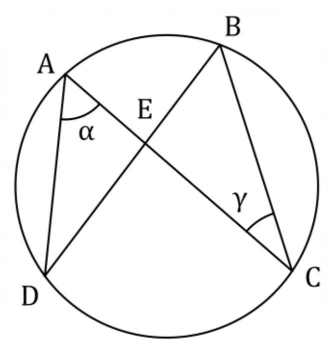
This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

(a) In the circle shown, chords AC and BD intersect at E, $\angle ADE = 27°$ and $\angle DEC = 84°$.

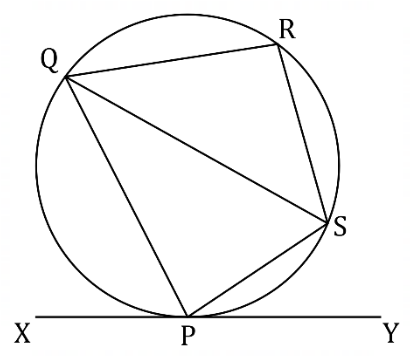
Determine the size of angles α and γ. (2 marks)



Solution
$\gamma = \angle ADE = 27°$ $\alpha = 84° - 27° = 57°$
Specific behaviours
✓ value of α ✓ value of γ

(b) In the circle below, $PQRS$ is a cyclic quadrilateral and XY is a tangent to the circle at P.

Given that $\angle XPQ = 72°$ and $\angle PQS = 53°$, determine $\angle QRS$. (3 marks)

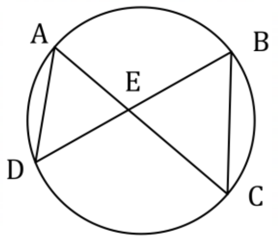


Solution
Alternate segment: $\angle SPY = \angle PQS = 53°$ Angle on a line: $\angle QPS = 180° - 72° - 53° = 55°$ Opposite angles: $\angle QRS = 180° - 55° = 125°$
Specific behaviours
✓ states $\angle SPY$ or $\angle QSP$ ✓ calculates $\angle QPS$ ✓ calculates $\angle QRS$

Question 2 **(6 marks)**

Points A, B, C and D lie on the circumference of a circle so that the chords AC and BD intersect at point E.

(a) Sketch a diagram to show triangle ADE and triangle BCE and prove that they are similar. (4 marks)

Solution

$\angle DAE = \angle CBE$ (angles stand on same arc)
$\angle AED = \angle BEC$ (vertically opposite angles)
Hence triangles are similar as three pairs of equal angles.
Specific behaviours
✓ neat, labelled sketch ✓ one pair of angles, with reason ✓ second pair of angles, with reason ✓ summary, using AAA reasoning

(b) In the case when the lengths of BE, CE and DE are 7 cm, 4 cm and 6 cm respectively, determine the length of AE. (2 marks)

Solution
Using intersecting chord theorem or similar triangles from (a): $AE \times 4 = 6 \times 7$ $AE = 10.5 \text{ cm}$
Specific behaviours
✓ indicates appropriate method ✓ calculates length

Question 3 **(6 marks)**

Let $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j}$.

(a) Determine $|2\mathbf{a} + 3\mathbf{b}|$. (3 marks)

Solution
$2\mathbf{a} + 3\mathbf{b} = 2\begin{pmatrix} 3 \\ -6 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 6 \\ -12 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ $= \begin{pmatrix} 3 \\ -9 \end{pmatrix}$ $\lvert 2\mathbf{a} + 3\mathbf{b} \rvert = \sqrt{90} = 3\sqrt{10}$
Specific behaviours
✓ calculates multiple ✓ calculates sum ✓ calculates magnitude

(b) Determine the vectors $\mathbf{m}$ and $\mathbf{n}$ given that $2\mathbf{a} + \mathbf{b} = \mathbf{m} - \mathbf{n}$ and $2\mathbf{b} - \mathbf{a} = 2\mathbf{m} + \mathbf{n}$. (3 marks)

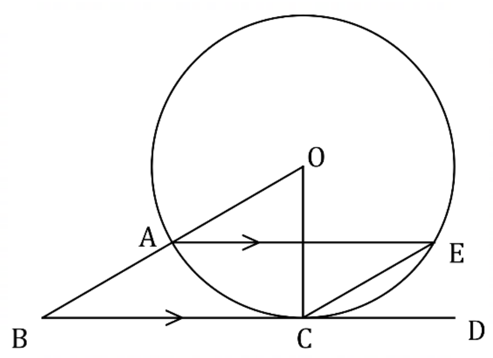
Solution
Adding equations to eliminate $\mathbf{n}$: $\mathbf{a} + 3\mathbf{b} = 3\mathbf{m}$ $\mathbf{m} = \frac{1}{3}\left(\begin{pmatrix} 3 \\ -6 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$ $= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\mathbf{n} = \mathbf{m} - 2\mathbf{a} - \mathbf{b}$ $= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} -5 \\ 10 \end{pmatrix}$
Specific behaviours
✓ eliminates one vector ✓ calculates $\mathbf{m}$ ✓ calculates $\mathbf{n}$

Question 4 **(4 marks)**

In the diagram, BD is a tangent at C to the circle with centre O, OB intersects the circle at A and chord AE is parallel to tangent BD.

Determine the size of $\angle ECD$ when the size of $\angle ABC = 38°$.

Justify your answer.



Solution
Using angle between radius and tangent property: $$\angle BOC = 90° - 38° = 52°$$ Using angle at centre and circumference on same arc property: $$\angle AEC = \frac{1}{2}(52°) = 26°$$ Using alternate angle property: $$\angle ECD = \angle AEC = 26°$$
Specific behaviours
✓ angle at centre ✓ angle on circumference ✓ justifies both above properties ✓ correct angle

Question 5 **(8 marks)**

(a) Determine the vector(s) that are parallel to $7\mathbf{i} + \mathbf{j}$ and have the same magnitude as $4\mathbf{i} - 3\mathbf{j}$. (4 marks)

Solution
Magnitudes:
$|(7,1)| = 5\sqrt{2}, \qquad |(4,-3)| = 5$
Unit vector in required direction:
$\hat{\mathbf{a}} = \frac{1}{5\sqrt{2}}(7,1)$
Hence vectors are:
$\pm\frac{5}{5\sqrt{2}}(7,1) = \frac{7\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}, -\frac{7\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$
Specific behaviours
✓ both magnitudes
✓ unit vector in required direction
✓ one required vector
✓ second vector in opposite direction

(b) Two vectors are $2\mathbf{i} + (2\lambda + 1)\mathbf{j}$ and $-7\mathbf{i} + (\lambda - 1)\mathbf{j}$, where λ is a constant.

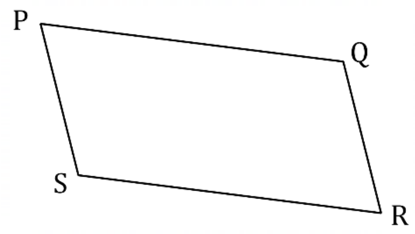
Determine the value(s) of λ so that the vectors are perpendicular. (4 marks)

Solution
Require scalar product to be zero:
$\begin{pmatrix} 2 \\ 2\lambda + 1 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ \lambda - 1 \end{pmatrix} = 0$
$-14 + (2\lambda + 1)(\lambda - 1) = 0$
$2\lambda^2 - \lambda - 15 = 0$
$(2\lambda + 5)(\lambda - 3) = 0$
$\lambda = -2.5, \lambda = 3$
Specific behaviours
✓ equates scalar product to zero
✓ calculates scalar product and expands
✓ simplifies and factorises
✓ states both values

Question 6 **(6 marks)**

The diagram shows parallelogram $PQRS$.

A theorem states that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.



(a) Complete the following expression of the theorem using vector notation: (1 mark)

$|\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2 =$

Solution
$\ldots = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 + |\overrightarrow{SR}|^2 + |\overrightarrow{PS}|^2$ *Allow* $\overrightarrow{PQ}$ *or* $\overrightarrow{QP}$ *or* $2|\overrightarrow{PQ}|^2$*, etc.*
Specific behaviours
✓ correct expression and vector notation

(b) Letting $\overrightarrow{PQ} = \mathbf{q}$ and $\overrightarrow{PS} = \mathbf{s}$, use a vector method to prove the theorem. (5 marks)

Solution
Note, for any vector $\mathbf{r}$: $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r}$. $LHS = |\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2$ $= |\mathbf{q}+\mathbf{s}|^2 + |\mathbf{q}-\mathbf{s}|^2$ $= (\mathbf{q}+\mathbf{s})\cdot(\mathbf{q}+\mathbf{s}) + (\mathbf{q}-\mathbf{s})\cdot(\mathbf{q}-\mathbf{s})$ $= \mathbf{q}\cdot\mathbf{q} + \mathbf{q}\cdot\mathbf{s} + \mathbf{s}\cdot\mathbf{q} + \mathbf{s}\cdot\mathbf{s} + \mathbf{q}\cdot\mathbf{q} - \mathbf{q}\cdot\mathbf{s} - \mathbf{s}\cdot\mathbf{q} + \mathbf{s}\cdot\mathbf{s}$ $= |\mathbf{q}|^2 + |\mathbf{s}|^2 + |\mathbf{q}|^2 + |\mathbf{s}|^2$ $= |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 + |\overrightarrow{SR}|^2 + |\overrightarrow{PS}|^2$ $= RHS$
Specific behaviours
✓ expresses $|\overrightarrow{PR}|$ and $|\overrightarrow{QS}|$ in terms of $\mathbf{q}$ and $\mathbf{s}$ ✓ uses scalar product to expand sums and differences   ✓ simplifies scalar products as magnitudes   ✓ expresses in terms of sides   ✓ logical presentation of proof using correct vector notation

Question 7 **(7 marks)**

Points A, B and C have position vectors $(-5, -4)$, $(7, 0)$ and $(-2, 7)$ respectively.

(a) Determine the position vector of M, the midpoint of A and B. (1 mark)

Solution
$\overrightarrow{OM} = \left(\frac{-5+7}{2}, \frac{-4+0}{2}\right) = (1, -2)$
Specific behaviours
✓ calculates position vector

(b) Determine the vectors $\overrightarrow{AB}$ and $\overrightarrow{CM}$. (1 mark)

Solution
$\overrightarrow{AB} = (7, 0) - (-5, -4) = (12, 4)$ $\overrightarrow{CM} = (1, -2) - (-2, 7) = (3, -9)$
Specific behaviours
✓ calculates vectors

(c) Show that $\overrightarrow{AB}$ and $\overrightarrow{CM}$ are perpendicular. (2 marks)

Solution
$\overrightarrow{AB} \cdot \overrightarrow{CM} = (12, 4) \cdot (3, -9) = 36 - 36 = 0$ Since scalar product is zero, then vectors are perpendicular.
Specific behaviours
✓ calculates scalar product ✓ interprets product

(d) Hence, or otherwise, determine the area of triangle ABC. (3 marks)

Solution
$\overrightarrow{AB}$ is base of triangle and $\overrightarrow{CM}$ is perpendicular height: $A = \frac{1}{2}\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{CM}\rvert$ $= \frac{1}{2} \times 4\lvert(3, 1)\rvert \times 3\lvert(1, -3)\rvert$ $= 6 \times \sqrt{10} \times \sqrt{10}$ $= 60 \text{ sq units}$
Specific behaviours
✓ identifies base and perpendicular height ✓ calculates magnitudes ✓ calculates area

Question 8 **(8 marks)**

(a) Show that $3\binom{5}{2} = 5\binom{4}{2}$. (2 marks)

Solution
$LHS = 3 \times 10 = 30, \quad RHS = 5 \times 6 = 30 \Rightarrow LHS = RHS$
Specific behaviours
✓ expresses LHS as product and evaluates ✓ expresses RHS as product and evaluates

(b) Show that $(n-k)\binom{n}{k} = n\binom{n-1}{k}$ for all n and k that are positive integers, $n > k$.

(4 marks)

Solution
$$\begin{aligned} LHS &= (n-k)\binom{n}{k} \\ &= \frac{(n-k)n!}{(n-k)!\,k!} \\ &= \frac{n!}{(n-k-1)!\,k!} \\ &= \frac{n(n-1)!}{([n-1]-k)!\,k!} \\ &= n\binom{n-1}{k} \\ &= RHS \end{aligned}$$
Specific behaviours
✓ expression using factorials for LHS ✓ divides fraction by $n-k$ ✓ factors n from numerator ✓ expresses as RHS

(c) Hence, or otherwise, evaluate $\binom{60}{55}$, given that $\binom{59}{55} = 455\,126$. (2 marks)

Solution
$$\begin{aligned} (60-55)\binom{60}{55} &= 60\binom{60-1}{55} \\ \binom{60}{56} &= 12 \times 455\,126 \\ &= 4\,551\,260 + 910\,252 \\ &= 5\,461\,512 \end{aligned}$$
Specific behaviours
✓ indicates appropriate method ✓ correct value

End of questions

Supplementary page

Question number: _________